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Supporting students' participation in authentic proof activities in computer supported collaborative learning (CSCL) environments

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Abstract In this paper, I review both mathematics education and CSCL literature and discuss how we can better take advantage of CSCL tools for developing mathematical proof skills. I introduce a model of proof in school mathematics that incorporates both empirical and deductive ways of knowing. I argue that two major forces have given rise to this conception of proving: a particular learning perspective promoted in reform documents and a genre of computer tools, namely dynamic geometry software, which affords this perspective of learning within the context of mathematical proof. Tracing the move from absolutism to fallibilism in the philosophy of mathematics, I highlight the vital role of community in the production of mathematical knowledge. This leads me to an examination of a certain CSCL tool whose design is guided by knowledge-building pedagogy. I argue that knowledge building is a suitable pedagogical approach for the proof model presented in this paper. Furthermore, I suggest software modifications that will better support learners' participation in authentic proof tasks.

Keywords Mathematical proof · NCTM reform · Dynamic geometry software · Knowledge forum

Proof is an important issue in mathematics education (Hanna 2000). Not only is it considered to be the essence of mathematics (Laborde 2000), but it also occupies a significant place in the recent reform documents of the National Council of Teachers of Mathematics (NCTM 2000). Traditionally, student proving activities have often been conceptualized one-dimensionally (both in curricula and in classroom practices) as providing a *deductive* argument. That is, students are either presented with or asked to write rigorous logical arguments to establish the truth of mathematical theorems. Therefore

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a certain view of mathematics is reinforced in schools: that one straightforwardly reaches a *true* conclusion in mathematics by starting with a set of axioms, the building blocks of any mathematical structure, and applying the rules of logic to those axioms. Empirical and inductive ways of knowing are almost always left out of the proving activity. Such a perspective does not create much room for students to view mathematics as a living body of knowledge that is produced collaboratively as a response and contribution to an evolving culture.

In this paper, I review both mathematics education and CSCL literature and discuss ways to improve CSCL tools to teach mathematical proof. More specifically, I introduce an emerging model for mathematical proof—geometrical proof in particular—for high school mathematics. This notion of proof incorporates both deductive and empirical ways of knowing, thus more closely reflecting expert mathematicians' proving activities. Such a model of proof gains momentum from the interaction of two forces: the NCTM-led reform that favors a *doing* perspective of mathematics, and the availability of dynamic geometry software (DGS), a genre of computer tools that allows experimentation and thus enables such a vision. Considering that mathematics is a product of collaborative work, I will then examine the potential of a certain CSCL environment, Knowledge Forum, and its guiding pedagogical approach, knowledge-building, to support the proof model presented in this paper. While I will highlight its promise, I will also suggest modifications in such software to make learners active participants in proving in CSCL environments.

Throughout this paper, by "deductive way of knowing" I refer to a certain way of justifying knowledge claims based on the logically connected sequence of assertions from accepted truths, which broadly include axioms, theorems, mathematical conventions, notions, methods, and definitions, for or against a mathematical statement (Stylianides and Silver 2004). An empirical way of knowing, on the other hand, is based on inductive, non-conclusive evidence for the truth of a mathematical statement (ibid.). In particular, this mainly includes checking a proper subset of all possible cases and using these to gain conviction for the validity of a mathematical claim. I use the expression 'proof model' to refer to the possible set of activities students might engage when working on a proof task—which is a *problem to prove* (Pólya 1981). Therefore a proof model might only constitute deductive or empirical ways of knowing. In this paper, however, I make the suggestion that it should integrate both.

I should also note that the proof model presented here is most relevant at the high school level. Although younger learners can build mathematically sound arguments (see Yackel and Hanna 2003), in high school the expectations are higher. It is at the high school level that NCTM expects students to understand and produce deductive arguments in forms that would be acceptable to professional mathematicians (NCTM 2000). Also, I particularly concentrate on geometrical proof, given that DGS largely targets geometry. While DGS could be used with other mathematical subjects, such as calculus or algebra, it mainly affords geometrical investigations.

Below I start by examining the nature of mathematical knowledge in the course of the philosophical shift from absolutism to fallibilism. Highlighting the essential role of community in the production and legitimization of mathematical knowledge, I point out that math education reformers in the U.S. recognize the changing assumptions about the nature of mathematical knowledge at the philosophical level. Following this is a view of school mathematics that emphasizes 'doing' mathematics rather than mastery of recorded knowledge. This perspective encourages a new dimension in learners' proving activity; learners can explore mathematical ideas through empirical exploration, which, even if theoretically desirable, had been not very feasible before the proliferation of computational technology in the classroom. The availability of DGS in classrooms, however, precipitated realization of the practicality of teaching a notion of proof that incorporates both empirical and deductive ways of knowing. I next turn to the prospects of CSCL software that affords authentic activity by setting community participation as the essential condition in the production knowledge. In particular, I examine the aptness of Knowledge Forum and its guiding theoretical background, knowledge-building pedagogy, for supporting the notion of proof presented. The unique nature of proving has also led me to offer technical modifications in order to enrich and support students' experiences with proof.

The nature of mathematical knowledge

For over 2,000 years, mathematics has been viewed as the source of infallible knowledge (Ernest 1998; Romberg 1992). The central activity through which mathematicians produce such knowledge is proving, which constitutes the means of justifying mathematical knowledge. As Davis and Hersh (1981) put it, for most "the name of the mathematics game is proof; no proof, no mathematics" (p. 147). The belief that mathematical knowledge is infallible arises from a certain conception of proof that Ernest (1998) describes as the following: "The idea underpinning the notion of proof is that of truth transmission. If the axioms adopted are taken to be true, and if the rules of inference infallibly transmit truth (i.e., true premises necessitate a true conclusion), then the theorem proved must also be true. For there is an unbroken and undiminished flow of truth from the axioms transmitted through the proof to the conclusion" (sic. p. 6). Lakatos characterized such a system as Euclidean (Rav 2007).

However, as Romberg observes, a growing number of philosophers of mathematics are challenging the perspective that mathematics is indubitable and infallible (Davis and Hersh 1981; Ernest 1998; Lakatos 1978; Rav 2007). Emphasizing the aspect of mathematics as the process of discovery, Lakatos showed that proof is not a procedure that enables the transmission of truth from assumptions to conclusions. Instead, "it means explanations, justifications, elaborations, which make the conjecture more plausible, more convincing, while it is being made more detailed and accurate under the pressure of counter-examples" (Davis and Hersh 1981, p. 347). While Lakatos applied his epistemological analysis to informal rather than formalized mathematics (ibid.), perhaps Rav (2007) presents a more radical argument when he says: "Mathematical proofs deploy genuine mathematical techniques not reducible to formal logic" (p. 26), arguing that even formalized mathematics may not be infallible. For Rav, mathematical reasoning does go beyond the rules of standard logic: mathematicians reason and make inferences based on meanings and informal notions of truth; therefore, there is no guarantee that proofs are flawless or without error.

At their core, these perspectives highlight the importance of the role of community in the production, and more importantly legitimization, of mathematical knowledge. The emphasis on community rather than on detached formal logic also explains why, at different periods in time, the notion of rigor in mathematics had different meanings. For Ernest (1998), the philosophy of Lakatos suggests that mathematics is fundamentally a community activity: "It suggests a pattern for the development of mathematical concepts, conjectures, proofs, and theories, as a collective enterprise and that it indicates the role and variety of interactions contributing to this development" (p. 128). Along with the work of individuals, the process of dialectical negotiation plays an essential role in the creation of mathematics.

Along the same lines, Livingston (1999, 2006) shows how the appearance of necessary truth or absolute certainty in mathematics can be examined as a cultural phenomenon, closely tied to the material aspects of the mathematical culture. This analysis leads to the idea that mathematicians are enculturated into the practices of proving, and the main difficulty for newcomers stems from the failure to easily adopt the mathematical discourse. His argument implies that we need to make novices into active participants, as only when they are brought into the mathematical culture will the act of proving make sense to them.

The view that mathematics is a process of inquiry, and that it is negotiations within the mathematical community that determine the legitimacy of mathematical knowledge rather than some unquestionable rules of logic, has important consequences for the teaching of mathematics (Romberg 1992), and therefore for proof. When this perspective is adopted, no longer can teachers present mathematical proof primarily from the deductive perspective, in which truth flows from axioms to conclusions through an unbreakable chain of reasoning.

The reformers in the U.S. seem to recognize this philosophical shift in mathematics from absolutism to fallibilism. The recent mathematics education reform movement led by NCTM (1989, 2000) first recognizes proof as the essence of mathematics and promotes a "doing" perspective of mathematics that underlines the dynamic and collaborative nature of mathematical knowledge. In doing so, the NCTM-led reform becomes one of the forces challenging the conception of students' proving activity as a merely deductive experience.

NCTM-led reform

Proof has a significant place in the recent reform movement in the United States (NCTM 2000). Although there was a time when proof had lost ground to heuristics both in the U.S. and U.K. (Hanna 2000), mathematical proof and reasoning is set forth as one of the five process standards in the NCTM reform document, Principles and Standards for School Mathematics (PSSM) (NCTM 2000). In this document proof is assigned the key role of promoting mathematical understanding in all mathematical content areas at all grade levels. PSSM recommends that instructional programs from prekindergarten through grade 12 should enable all students to:

> [r]ecognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof (NCTM 2000, p. 56).

Not only does NCTM assign a key role to proof in mathematics learning, it also promotes a fallibilist perspective about mathematics (Romberg 1992). In contrast to the absolutist view, fallibilism rejects the notion that mathematics is a finished product or a fixed body of knowledge to be transmitted and mastered by students. In NCTM documents, "'[k]nowing' mathematics is 'doing' mathematics. A person gathers, discovers, or creates knowledge in the course of some activity having a purpose" (NCTM 1989, p. 7). This process and meaning-centered perspective has led many researchers to think about what makes proof *meaningful* and thus to reexamine the nature of mathematicians' proving activity.

We know that students have many difficulties with proof (Senk 1985; Silver and Carpenter 1989; Usiskin 1987). In discussing the ways to remedy the problem of teaching and learning to prove, Senk (1985) emphasizes that proving needs to be made meaningful for pupils so that they understand and appreciate mathematical proof the way mathematicians do. The same recommendation is also made by others. Reiss and Renkl argue that the belief that a mathematical proof is a linear, systematic and formal sequence of steps is one important factor that impedes learning. Citing Boero (1999), they point out that the expert act of proving is not based on a linear model. Rather than following a straight logical argumentation, mathematicians continuously move between explorative, inductive, and deductive processes.

Moreover, for mathematicians proof is not necessarily a prerequisite for conviction; mathematicians are usually convinced about the "truth" of the theorem prior to proving it (de Villiers 1998). In mathematical research, "[C]onviction often precedes the actual proof and is probably far more frequently a prerequisite for finding a proof" (de Villiers 1997). As Davis and Hersh (1981) state, the reason mathematicians want a proof for the famous Riemann Hypothesis is that they know from convincing heuristic reasoning that it is true but seek to understand *why* it is true. Thus investigation and heuristic explorations are parts of the proving process.

As a result, there is a growing emphasis on the role of empirical explorations in mathematics education. Hoyles (1997) observes that researchers argue that students should have opportunities to test and refine their conjectures, which will give them conviction of their conjectures' validity. For Edwards (1997) the phase of investigations/explorations can be considered the conceptual territory before proof. It is a "space" of potential precursors to proof that includes "[w]ays of thinking, talking, and acting that support the goal of seeking and establishing mathematical certainty" (ibid., p. 189). Although she cautions that engagement with exploration activities does not necessarily lead to the construction of a formal rigorous proof, they can nevertheless provide the basis for a richer understanding of a proof. In her study, she argues that although students were not able to create proofs of their own, they were knowledgeable enough to understand the proof offered by the investigator after working with a microworld where they had the chance to explore the Euclidean transformations of planes.

In summary, mathematical learning that emphasizes a *doing* perspective tends to favor experimentation and exploration of mathematical ideas. While earlier conceptions of proof excluded empirical ways of knowing within proving activity, new ideas about the nature of learning have proposed the opposite. Researchers have come to believe that exploration/experimentation of mathematical ideas is an important ingredient of proving activity, for it reflects the work of expert mathematicians, gives students the opportunity to work from their own intuitions and investigations, and thus potentially makes proving more meaningful and accessible.

The role of dynamic geometry software (DGS)

The realization of the aforementioned vision of proving has been enabled by the immediate availability of a very powerful set of computer tools, namely dynamic geometry software (DGS), in classrooms. DGS materializes the vision described above, since it facilitates heuristic explorations. Experimentation is now possible in mathematics classes due to two main features of DGS: the ability to drag objects to manipulate them dynamically, and visual control.

The defining feature of DGS is the continuous real-time transformation called 'dragging.' When the elements of a drawing are moved, this feature allows the construction to respond dynamically to the altered conditions (Goldenberg and Cuoco 1998) by maintaining the invariant. This aspect of DGS facilitates conjecturing and more inductive approaches to geometric knowledge, as students can reason about the generality of their hypotheses for several cases (Kaput 1992).

Another distinguishing aspect of DGS is its visual control in contrast to the symbolic control required in programming environments such as LOGO (Healy and Hoyles 2001). For some, the ability to program a computer is a valuable educational activity, for it will provide the foundational/intuitional skills to learn mathematics and science easily (di Sessa 2000; Papert 1980). However, as Norman's (1994) famous Hanoi Puzzle example illustrates, we humans are better at operating by perceptual routines. In that regard, DGS can be considered a more accessible medium through which one can express mathematical ideas and explore geometrical relations.

To better understand how DGS affords an empirical way of knowing, consider the example given below. In the first figure, the medians of triangle ABC were constructed in the Geometer's Sketchpad (Jackiw 1995, 2001), one of the most well known DGS applications (Fig. 1). Students can see that the medians concur at point G. Figures 2 and 3 were captured by dragging point C to different places. Although it is not possible to represent the real-time transformation on paper, the reader can imagine that medians keep intersecting at one point as one changes the shape of the triangle ABC by dragging its vertices.

Furthermore, use of DGS does not simply make the task more efficient; it also has more educational value. DGS constructions differ from drawings in that they are based on theory and their production requires theoretical geometry knowledge. For this reason, some tasks are modified when working with DGS (Laborde 2001). These are the tasks for which the DGS environment deeply affects the solving strategy. Laborde (ibid.) gives the example of constructing a square using DGS versus squared paper. With squared paper such construction consists essentially of drawing four sides of equal lengths along the lines of

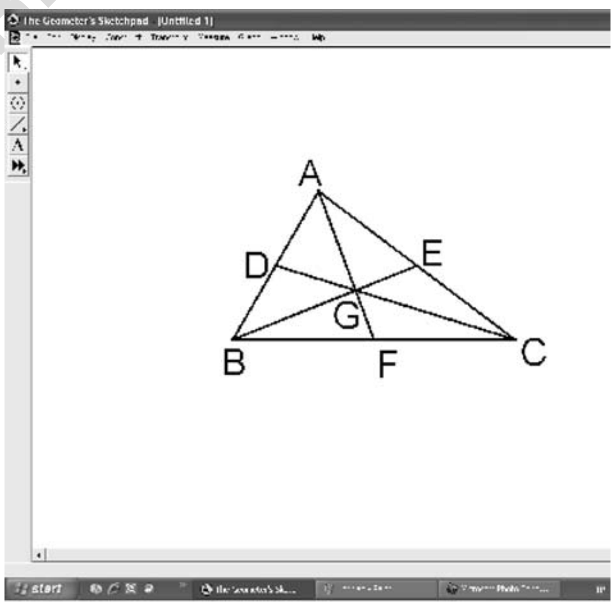

Fig. 1 Triangle ABC and its medians constructed using the Geometer's Sketchpad

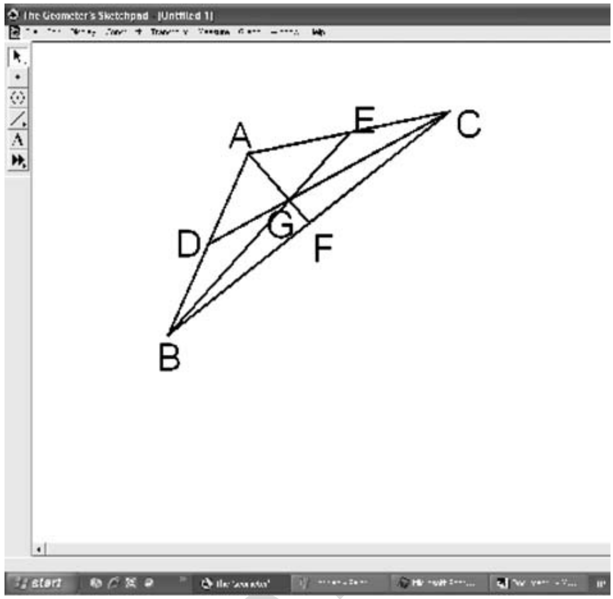

Fig. 2 Same construction, vertex C is dragged

the paper. While this is achieved by following the lines of the paper and counting the squares, which is controlled by perception, the same task in DGS must be done by satisfying both the perpendicularity and the congruence of sides. The congruence cannot be obtained perceptually by counting, but uses a circle as a tool for transforming a given distance. Therefore the task with DGS requires more mathematical knowledge about the properties of a square and the characteristic property of a circle.

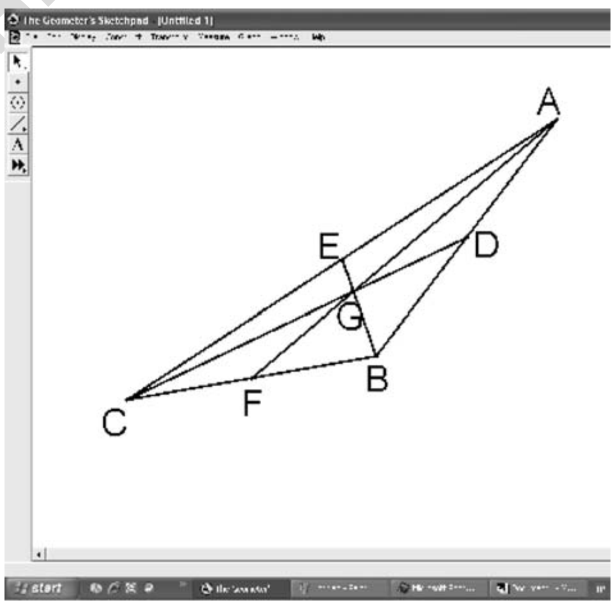

Fig. 3 Same construction, vertex C is dragged some place else

Conceptualizing proving activities

Emanating from the interaction between the ideas of reform and the immediate availability of computational technology in classrooms is a notion of proof that encompasses both empirical and deductive ways of knowing. Structuring students' proving activities along both dimensions will give students the opportunity to experience mathematics as a process of inquiry, not a finished product that needs to be mastered. DGS affords an environment in which students can work with hunches and gain the confidence necessary to construct proofs for their conjectures. This perspective also implies that students should be initiated into a discourse in which *inventing* proof problems is as valuable as solving them. As Pólya (1945) states, if a student never has the chance to solve a problem s/he invents, her/his mathematical experience is not complete.

In addition, this conception of proof will provide the opportunity to emphasize the *explanation* function of proof (that is, providing insight into why the theorem is true) as a viable alternative to the notion of proof as verification presented in the traditional teaching approach (de Villiers 2003; Hanna 2000). While proof has several functions in mathematics,[1] for Hanna, the explanation function is the most important one in the educational domain.

Knowledge-building pedagogy as a test case

How, then, should new technologies be utilized to support both empirical investigations and deductive reasoning while "doing" mathematics as a community product? Following Lipponen (2002), the computer can afford students such engagement with proof in at least two different ways. First, students' collaborative activities can be supported *around* the computer in a face-to-face setting. Here the computer mainly acts as a referential anchor mediating the coordination of collaborative actions. The contributions made can be observed and structured by the teacher. The second possibility is to use a networked learning environment to structure the collaboration. This second use involves significant innovation, especially for those who believe that educational technology needs to broaden the technology repertoire beyond classroom-oriented tools and exploit networking infrastructures in order to maximize learning opportunities (Pea et al. 1999). The use of CSCL software generally belongs to the second category.

Among the software tools included in the genre of CSCL software, Knowledge Forum, introduced by Scardamalia and Bereiter (2003, 2004), deserves special attention, since its main goal is to initiate learners into a knowledge-creating culture. The design of Knowledge Forum is guided by knowledge-building pedagogy, which aims to support practices that are geared towards building knowledge. From the fallibilist point of view, this perspective appears to be a fitting pedagogical model for learners' creation of new mathematical knowledge.

Knowledge-building pedagogy highlights "knowledge advancement as idea improvement rather than as progress toward true or warranted belief" (Scardamalia and Bereiter

[1] Along with *verification*, these functions include *explanation* (providing insight into why a mathematical statement is true), *discovery* (the discovery or invention of new results), and *communication* (the negotiation of meaning), *intellectual challenge* (the self-realization/fulfillment derived from constructing a proof), and *systematization* (the organization of various results into a deductive system of axioms, concepts and theorems) (de Villiers 2003).

2004). This idea underlies the belief that authentic knowledge-building work could take place in classrooms. Idea improvement is by far the most essential characteristic of knowledge-building work carried out at professional levels—and just like professionals, learners can create mathematics that is in constant production.

As mentioned, this pedagogy also guided the design of computer tools. Knowledge Forum is second generation software[2] aimed at bringing school activity into a closer alignment with the processes by which knowledge advances in the disciplines (Scardamalia and Bereiter 2004). To this end, Knowledge Forum has a number of features that afford students' knowledge-building within a community. An asynchronous discussion environment that allows reflection and revision appears to be an especially important characteristic in increasing the authenticity of student activity. Not only can users post content with notes, but they can also organize the content to add structure to it. In addition, Knowledge Forum scaffolds learning by structuring contributions students can make. Some of the theory-building scaffolds are "my theory," "new information," "this theory explains that," etc.

Although knowledge-building pedagogy and its accompanying software appear very likely to promote a "doing" perspective of mathematics, not many studies report the use of Knowledge Forum in mathematics learning. Those studies that do exist suggest that CSCL software can support a "doing" perspective of mathematics as envisioned by the reform in math classes as well.

One study reporting successful use of CSCL software in mathematics classrooms was conducted by Nason and Woodruff (2003). They argued that using Knowledge Forum along with the model-eliciting problem solving approach enhanced the authenticity of classroom mathematical activity. They pointed out the difference between school mathematics activities and those of mathematicians: while mathematicians often form research communities to produce and improve mathematical conceptual artifacts, school mathematics mostly focuses on individual completion of worksheets and textbook exercises. With students engaged in model-eliciting problem solving within Knowledge Forum, the researchers observed that students' activity had much in common with many mathematical research communities in regards to symbolizing, communicating, mathematizing, and collective understanding. In addition, at the beginning of the study most students had absolutist perspectives about mathematics. However, by the end of the study, mathematics was viewed as the outcome of social processes and mathematical knowledge as fallible and eternally open to revision.

Nason and Woodruff's (2003) work emphasized that the nature of mathematical problems used in CSCL mattered in establishing and maintaining an authentic knowledge-building community. It also showed that CSCL software has effectively supported collaborative knowledge-building in mathematics classrooms. They attributed this success to the features of Knowledge Forum: an asynchronous discussion environment that encourages reflection and revision by removing time and place constraints, and the scaffolds built into the software that structure the discourse.

Another successful implementation of CSCL software in mathematics classrooms was reported by Moss and Beatty (2006). Their purpose was also to investigate whether Knowledge Forum could promote inquiry orientation in mathematics, as it had been observed to do in science classes. More specifically, they looked at how, working collaboratively, students managed to find algebraic rules for mathematical generalization problems.

Upon starting their research project, Moss and Beatty (2006) believed that a collaborative environment supported by computers would provide an authentic platform

[2] The first generation was Computer Supported Intentional Learning Environments (CSILE).

for student participation for solving generalization problems. They found that students both discovered and revised multiple rules for these problems. Like Nason and Woodruff (2003), they thought that idea development and knowledge-building were supported by certain features of the software: the asynchronicity of the interaction environment, which gives extended time to think, and a structure that allows revision at any time.

These studies suggest that students can be led into meaningful proof activity in CSCL environments. The pedagogy behind Knowledge Forum in particular parallels math reformers' proposals for school mathematics. Further, the research suggests that the features of Knowledge Forum, such as the asynchronous discussion environment and the availability of scaffolds structuring discourse, can also support types of activity that resemble the activity of professional mathematicians more closely. However, some technical modifications might be necessary in order to make a knowledge-building environment useful for the proof model developed in this paper.

Moss and Beatty's (2006) findings suggest that the text-based discourse space has benefited students, since, as these researchers believed, students made more efforts to communicate their ideas, and the need to clearly communicate them enhanced students' mathematical understanding. In the case of proof, however, I propose that the text-based character of such environments needs to be supported with a DGS component to enable students to generate, test, and share their conjectures.

In a paper written in 1999 describing the research agenda for the Center for Innovative Learning Technologies (CILT), Pea et al. (1999) set an expectation for the upcoming learning technologies: "We anticipate new technologies supporting highly interactive learning conversation, mediated by complex symbolic representations, such as mathematical notations, scientific visualizations, and multimedia case studies. These technologies will draw jointly on powerful modeling tools and participants' informal sketches and annotations" (p. 31). In also setting it as a research agenda, they particularly highlight the importance of the shared active representations in science and mathematics that afford successful learning conversations online or locally. Although this category includes simulations, visualizations, mathematical notations, text, graphs and so on, the specific emphasis was on active simulations that can be jointly controlled by remote participants. Pea et al. summarize the importance of such shared representations as creating the common ground for conversational support and helping to make sense of a given problem together.

The proposal made here parallels this idea. Collaborative environments aimed at supporting students' proving activity, for geometry proofs in particular, should include a DGS component. More specifically, a CSCL tool that could enable students to participate meaningfully in building mathematical arguments should provide a shared whiteboard on which students can easily and mutually create, edit, and display dynamic constructions, linked with equations and graphs. In fact, research in CSCL that focuses on building tools for shared referencing highlights the importance of such environments (Stahl 2006).

The need to support proof activity for learners

However, it would be naïve to think that simply integrating DGS into students' proving activities will result in students working like mathematicians. Many researchers have raised concerns that students using DGS could be misconceptualizing the nature of mathematical truth; that is, coming to believe that a confirmation of a conjecture for several cases would secure its truth (Allen 1996; Chazan 1993b; Hoyles and Jones 1998). Hoyles and Jones caution that when pupils can generate their own empirical evidence, this means that they have little chance to appreciate the importance of logical argument and to produce a proof. They point to the danger of DGS; that it may limit the mathematical work of the majority to empirical argument and pattern-spotting. As conviction can be obtained easily by dragging, DGS environments may also prevent students from understanding the need and function of proof (Hadas et al. 2000).

One of the strongest critics of DGS is Allen (1996). He urges caution in the use of DGS based on the argument that DGS tends to blur the distinction between illustration and proof. He gives the example of a dissection proof of the Pythagorean Theorem in which two smaller squares are partitioned in such a way that their parts can be reassembled to cover the largest square. Since no previous theorems are invoked, such an illustration cannot be considered proof. Furthermore, Allen claims that DGS makes little contribution to the analysis of construction problems in Plane Geometry, and none at all to the proof necessary to show that the construction is correct.

Obviously, while DGS enables a crucial dimension of more *meaningful* proving activity (exploration and experimentation), the very same aspect that affords experimentation (dragging) appears to be the source of concern if it is also used for *justification*, that is, to validate the truth of a mathematical claim. How could designers eliminate this drawback?

Hoyles and Jones (1998) note that there is already a noticeable trend toward using DGS to identify patterns, generate cases, measure lengths and angles, etc.—in other words, simply to provide data. Their concern is that this data-driven approach could cause students to side-step all the important mathematical content, if we are not careful in our attempts to incorporate powerful technologies into mathematics teaching and learning. In developing DGS activities that lead to meaningful experiences with proof, it is essential that we encourage learners to make conjectures about the relationships between geometrical objects, and that we design activities in such a way that students can link empirical and deductive reasoning throughout the mathematical activity.

Similarly, Reiss and Renkl (2002) emphasize that as exploratory processes are an important part of mathematical activity, learning environments should both provide opportunities for individual exploration and foster mathematical argumentation. Yet empirical exploration of a problem does not necessarily lead students to mathematical proof. Thus, "[a] learning environment for proof and argumentation should not only allow for exploration and investigations but should also provide specific help and support with respect to the proving process" (ibid., p. 30). As Hanna (2000) makes clear, there are two reasons for this: first, certainty in mathematics is only achieved by proof; second, it is the responsibility of educators to teach that reason to our students—that is, formulating and testing conjectures does not constitute proof.

Furinghetti et al. (2001) liken this approach in mathematics, which promotes transition from empirical exploration to theoretical thinking, to experimental disciplines. They say the analogy lies in the fact that, in both cases, the starting point is observation and exploration in which regularities are discovered, and only later are convincing arguments provided in order to validate or refute these conjectures. The transition to theoretical thinking, however, "[r]equires reasoning within a theory and producing arguments which must be valid within that theory" (p. 324).

Empirical studies conducted in classroom settings also support these insights by highlighting the role of 'social infrastructure' (Bielaczyc 2001) in realizing the potential of DGS as a useful tool for teaching proof. More specifically, with carefully designed tasks, in a classroom environment that supports conjecturing and deductive justifications, along with teacher guidance, students can benefit from the DGS environment in developing proving skills (Hadas et al. 2000; Healy and Hoyles 2001; Jones 2000; Mariotti 2000, 2001;

Marrades and Gutierrez 2000). All these studies lead to the conclusion that in learning environments adopting DGS, students should be explicitly asked for and guided towards deductive arguments after their explorations in order to avoid teaching misconceptions about mathematical truth. Therefore the tools that aim to support proving should be structured in a way that encourages students to build deductive arguments after they generate and test their ideas empirically.

Scaffolding proof tasks

So far one of the main ideas I have highlighted is that technology provides the opportunity to initiate learners into a knowledge-building discourse in mathematics. Meanwhile, the software tools can also be designed to provide support for the challenging aspects of the subject matter (Reiser 2004). This kind of support is typically captured by the notion of ‘scaffolding,’ which originates in Vygotksy’s work. With scaffolding learners can complete complex tasks that would be otherwise beyond their reach.

Reiser (2004) states that one way tools can help learners with difficult tasks is they can distribute work and reduce what is expected from learners. For instance, calculators offload computation and allow learners to focus on the conceptual aspects of a task. Meanwhile, tools can also transform tasks. This idea is generally captured by the notion of the ‘reorganizing’ nature of tools (Pea 1985). In mathematics education research, there is a growing recognition of the “reorganizer” nature of DGS. That is, several authors point out that DGS does not simply make the task more efficient; rather, it fundamentally changes the task (Healy and Hoyles 2001; Jones 2000; Lerman 2001). Within the DGS learning environment, “[t]he computer is more than a mediating bridge, as its function cannot be simply reduced to a learning aid—to be discarded after the concepts and procedures have been acquired” (Holzl 2001, p. 81).

Therefore, the design of the tool directly influences the task. Reiser (2004) thinks this is an advantage for instructional designers, since they can shape the task for learners through the design of the tools. Reiser offers two mechanisms of scaffolding in software tools. These tools can *structure* the task and they can *problematize* subject matter by causing students to consider aspects of the task that they might otherwise overlook.

Reiser’s concept of structuring includes decomposing the task, focusing effort, and monitoring. The software can decompose a complex task by designating the necessary actions to take and their order. Such systems generally use checklists or diagrams to help learners identify and implement the processes they are to perform. The software tools can also help learners focus on the aspects of the task that are more productive for learning. One way to achieve this is by helping learners work together more effectively. Keeping track of plans and monitoring progress is another support mechanism software tools can provide. Prompts, agendas, or graphical organizers can remind learners of important goals and criteria to apply to their work.

Another mechanism for scaffolding is making some aspects of student work “problematic.” Instead of simplifying the task, the software can lead learners to confront the complex nature of the task. Problematizing involves focusing students’ attention on an aspect of a situation that needs resolution, which might be achieved by creating a sense of dissonance or curiosity. It also requires an affective component. This involves eliciting student commitment to the extent that they can fully engage with the task. Reiser (2004) states that “the notion of problematizing goes beyond avoiding giving too much help or fading the help as learners develop increasing expertise. This core of this approach is to guide the learner into facing complexity in the domain that will be productive for learning, for example, by connecting their work on a problem to disciplinary frameworks" (p. 288). Therefore, focusing on the problematizing potential of software tools can help address the difficulties students have with proof.

And indeed, students have many difficulties with proof. High school students perform poorly on proof-related tasks (Senk 1985; Silver and Carpenter 1989; Usiskin 1987) and demonstrate limited understanding of mathematical reasoning and proof (Chazan 1993a; Coe and Ruthven 1994). Coe and Ruthven found that students' proof strategies were primarily and predominantly empirical. They observed that the way students validated a conjecture mostly took the form of testing it against a few examples. Chazan summarized research findings on students' views of empirical and deductive argumentation into two categories. The first is the students' belief that empirical evidence counts as proof. Some students are content that acts that produce empirical data, such as measuring, can allow one to reach certain conclusions in mathematics. The second finding is that students believe that deductive proof is simply evidence. In other words, they believe that deductive proof is not sufficient to secure the truth of the statement; additional checks will be necessary to secure that truth. As discussed earlier, DGS might just aggravate these difficulties by functioning as a powerful medium for granting the truth of a mathematical statement.

Problematizing tools can address learner difficulties by eliciting articulation, eliciting decisions, and bringing gaps and disagreements to the surface. Eliciting articulation involves the ways in which the software tool restricts student contributions. The theory-building scaffolds of Knowledge Forum, such as "my theory" and "new information," fall into this category. Software that elicits decisions similarly may require explicit representations more precise than natural language or students' paper and pencil work. That way it might enable learners to make critical decisions, such as classifying the way evidence connects to positions in an argument. Reiser (2004) states that such constrained representations that are created and shared collectively eventually become the reference for further identifying issues that need resolution. Bringing disagreements to the surface can be vital to achieving disciplinary goals.

Following Reiser's (2004) thinking, one can conclude that Knowledge Forum's scaffolds can also act as problematizing mechanisms, and this has great potential. Other examples include software such as Galapagos Finches (Reiser et al. 2001) and Animal Landlord (Smith and Reiser 1998). Galapagos Finches forces students to express their hypotheses within a scientific theory, such as natural selection. Animal Landlord asks students to organize their analyses into 'observations' and 'interpretations' and thus reinforces a key epistemic distinction in scientific practice. This can also be achieved in the case of proof. The software can ask students to sort their arguments into two kinds: *conviction* (*gained by empirical investigation*) and (*deductive*) *proof*. Such an explicit distinction can also provoke discussion addressing the interrelation between the two. Furthermore, the software must reinforce the need for proof in order to grant the truth of a mathematical statement. Technically such software can make use of scripts to achieve this end. The success of scripts in areas such as argumentation (Stegmann et al. 2007) suggests that this technique can also be effectively applied to foster proof skills by addressing both aspects of the proof model.

Conclusions

In his seminal work *How to Solve It*, Pólya (1945) identifies two faces of mathematics: one that is deductive science, which is presented in a Euclidean sense, and another that is experimental and inductive activity, which characterizes mathematics in the state of becoming. Pólya states that both are as old as the discipline itself. Yet, in school mathematics the dominant pedagogical approach has been the deductive face of mathematical knowledge; this has especially been the case for proof.

However, the recent math education reform in the U.S. and the availability of DGS pose challenges for this conception of proof in high school mathematics. The reform movement recognizes the philosophical shift in the discipline of mathematics and proposes a model of learning that closely represents the work of professional mathematicians. In this paper, I suggest that in the case of proof, this entails incorporating both empirical and deductive ways of knowing into school learning. By affording empirical investigations, DGS makes the implementation of this perspective possible in classrooms.

Work in learning sciences also converges upon the general idea that knowledge is a human construction. Knowledge-building pedagogy is one attempt aimed at translating knowledge creation as it is handled in larger society into school settings. This pedagogical model also informs the design of software tools that aim to support knowledge-building practices. Knowledge-building pedagogy and its accompanying software tools hold great undeveloped promise for the proof model described in this paper in particular and for mathematical learning in general.

One premise of this model is that learning is an integral part of the knowledge-building process. That is, students learn by engaging in knowledge creation practices. The design of software is guided by this principle. An asynchronous learning environment, the ability to structure contributions, and the features that can direct discourse (scaffolds) are all geared towards supporting knowledge-building.

However, along with emphasizing the potential of knowledge-building and its software for mathematical learning, I have suggested some modifications in this or similar software tools to better support students' participation in proof tasks. The first suggestion was to integrate DGS into a networked knowledge-building environment, where students can test their hypotheses and share their findings. Although browser-based technology still poses constraints to the mutual manipulation of dynamic representations, this would be a worthwhile endeavor.

At the same time, I also cautioned against the idea that simply integrating DGS into a knowledge-building environment will be sufficient for teaching learners to work like mathematicians. Yet following Reiser (2004), I suggested that building 'problematizing' mechanisms into the software might help learners to map their strategies onto important disciplinary distinctions. For the proof model presented in this paper, this can be achieved by using a script that explicitly requires learners to sort their ideas into two categories: empirical conviction or deductive proof. In addition, the script should also require students to build deductive arguments after they generate and test their ideas in an empirical way.

However, are there any contradictions between this level of structuring and the theory behind knowledge-building pedagogy? In other words, is there a conflict between what needs to be transmitted to students and the processes that make them active constructors of knowledge? The principles behind the knowledge-building pedagogy suggest that the efforts guided by the principle of advancing knowledge may not always overlap with the efforts to cover the whole curriculum.

Scardamalia and Bereiter (2004) introduce the notion of "epistemic artifacts" to capture the idea that members use both abstract (theoretical) and concrete models in order to advance knowledge. The creation of further knowledge is possible with these epistemic artifacts. They say that the value of epistemic artifacts is judged by the extent to which they enable the further growth of knowledge, rather than by their conformity to accepted knowledge. Along with an emphasis on advancement of knowledge, Scardamalia and Bereiter (2003) also highlight the distributed nature of knowledge-building work: “An optimal KBE (knowledge building environment) will exploit the fullest possible potential of ideas to be improved by situating them in worlds beyond the minds of their creators and compounding their value so that collective achievements exceed individual contributions” (p. 270). It seems that these ideas put emphasis on learning as ‘participation’ rather than learning as ‘acquisition’ of pre-determined facts (Sfard 1998). Emphasis on the distributed view of knowledge and endless process for idea improvement in its strongest form might go against the long-established ideals of education. Yet what Scardamalia and Bereiter (2004) say is students are legitimate participants:

> At the deepest level, knowledge building can only succeed if teachers believe students are capable of it. This requires more than a belief that students can carry out actions similar to those in knowledge creating organizations and disciplines. It requires a belief that students can deliberately create knowledge that is useful to their community in further knowledge building and that is a legitimate part of the civilization-wide effort to advance knowledge frontiers (p. 26).

The dilemma surfaces: children are in need of mastering necessary knowledge and it is through education that they become ready for the actual work later in their lives. Though this dilemma is perhaps the main reason approaches like the one proposed in this paper frequently remain peripheral in education, rather than taking center stage, the dilemma can equally function as a constructive limitation to further advance our own conceptions of education.

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